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Uniform Parameterisation of Phase Based Cooperations

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Uniform parameterisations of phase based cooperations are defined in terms of formal language theory. For such systems of cooperations a kind of self-similarity is formalised. Based on deterministic computations in shuffle automata a sufficient condition for self-similarity is given. Under certain regularity restrictions this condition can be verified by a semi-algorithm.

Key Words: cooperations as prefix closed languages, abstractions of system behaviour, self-similarity in systems of cooperations, iterated shuffle products, deterministic computations in shuffle automata

1. INTRODUCTION

As an example for cooperations let us consider an e-commerce protocol, that determines how two cooperation partners have to perform a certain kind of financial transactions. As such a protocol should work for several partners in the same manner, it is parameterised by the partners and the parameterisation should be uniform w.r.t. the partners.

To be able to verify entire families of parameterised systems, independent of the exact number of replicated components, in [Ochsenschläger and Rieke 2007] we developed an *abstraction based approach* to extend our current tool supported verification techniques to such systems.

In this paper (Sect. 2) we formalise uniform parameterisations of two-sided cooperations in terms of formal language theory, such that each pair of partners cooperate in the same manner, and that the mechanism (schedule) to determine how one partner may be involved in several cooperations, is the same for each partner. Generalising each pair of partners cooperating in the same manner, the following kind of self-similarity is desirable for such systems of cooperations: From an abstracting point of view, where only actions of some selected partners are considered, the complex system of all partners behaves like the smaller subsystem of the selected partners.

The main goal of this paper is a sufficient condition for this self-similarity (Sect. 6). The main concepts for such a condition are structuring schedules into phases, which may be shuffled in a restricted manner (Sect. 3), and shuffle automata, whose deterministic computations unambiguously describe how a cooperation partner is involved in several phases (Sect. 4 and 5). For the notion of self-similarity it is of interest to know, which kind of dynamic system properties are compatible with our notion of abstraction. This is discussed in Sect. 6.

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In [Jantzen 1985] the operations shuffle and iterated shuffle and in [Jedrzejowicz 1999] and [Jedrzejowicz and Szepietowski 2001] structural properties of shuffle automata are analysed and an algorithm for assigning a shuffle expression denoting the language of the automaton is given. In [Björklund and Bojanczyk 2007] the close connection between shuffle expressions and multicounter automata is demonstrated.

Examples for the theory developed will be given in a forthcoming paper.

2. PARAMETERISED COOPERATIONS

The behaviour L of a discrete system can be formally described by the set of its possible sequences of actions. Therefore $L \subset \Sigma^*$ holds where Σ is the set of all actions of the system, and Σ^* (free monoid over Σ) is the set of all finite sequences of elements of Σ, including the empty sequence denoted by ε. This terminology originates from the theory of formal languages [Sakarovitch 2009], where Σ is called the alphabet (not necessarily finite), the elements of Σ are called letters, the elements of Σ^* are referred to as words and the subsets of Σ^* as formal languages. Words can be composed: if u and v are words, then uv is also a word. This operation is called the *concatenation*; especially $\varepsilon u = u\varepsilon = u$. A word u is called a *prefix* of a word v if there is a word x such that $v = ux$. The set of all prefixes of a word u is denoted by $\text{pre}(u)$; $\varepsilon \in \text{pre}(u)$ holds for every word u.

Formal languages which describe system behaviour have the characteristic that $\text{pre}(u) \subset L$ holds for every word $u \in L$. Such languages are called *prefix closed*. System behaviour is thus described by prefix closed formal languages.

Different formal models of the same system are partially ordered with respect to different levels of abstraction. Formally, abstractions are described by so called alphabetic language homomorphisms. These are mappings $h^* : \Sigma^* \longrightarrow \Sigma'^*$ with $h^*(xy) = h^*(x)h^*(y)$, $h^*(\varepsilon) = \varepsilon$ and $h^*(\Sigma) \subset \Sigma' \cup \{\varepsilon\}$. So they are uniquely defined by corresponding mappings $h : \Sigma \longrightarrow \Sigma' \cup \{\varepsilon\}$. In the following we denote both the mapping h and the homomorphism h^* by h. In this paper we consider a lot of alphabetic language homomorphisms. So for simplicity we tacitly assume that a mapping between free monoids is an alphabetic language homomorphism if nothing contrary is stated.

To describe a two-sided cooperation, let $\Sigma = \Phi \mathbin{\dot\cup} \Gamma$ where Φ is the set of actions of cooperation partner F and Γ is the set of actions of cooperation partner G. Now a prefix closed language $L \subset (\Phi \mathbin{\dot\cup} \Gamma)^*$ formally defines a two-sided cooperation.

For parameter sets I, K and $(i,k) \in I \times K$ let Σ_{ik} denote pairwise disjoint copies of Σ. The elements of Σ_{ik} are denoted by a_{ik} and $\Sigma_{IK} := \dot{\bigcup}_{(i,k)\in I\times K} \Sigma_{ik}$. The index ik describes the bijection $a \leftrightarrow a_{ik}$ for $a \in \Sigma$ and $a_{ik} \in \Sigma_{ik}$. Now $\mathcal{L}_{IK} \subset \Sigma_{IK}^*$ (prefix-closed) describes a *parameterised cooperation*. To avoid pathological cases we generally assume parameter and index sets to be non empty.

For $(i,k) \in I \times K$ let

$$\pi_{ik}^{IK} : \Sigma_{IK}^* \to \Sigma^* \text{ with } \pi_{ik}^{IK}(a_{rs}) = \begin{cases} a \mid a_{rs} \in \Sigma_{ik} \\ \varepsilon \mid a_{rs} \in \Sigma_{IK} \setminus \Sigma_{ik} \end{cases}$$

For uniformly parameterised systems $\mathcal{L}_{IK}$ we generally want to have

$$\mathcal{L}_{IK} \subset \bigcap_{(i,k)\in I\times K} ((\pi_{ik}^{IK})^{-1}(L)).$$

In addition to this inclusion $\mathcal{L}_{IK}$ is defined by *local schedules* that determine how each "version of a partner" can participate in "different cooperations". More precisely, let $SF \subset \Phi^*$, $SG \subset \Gamma^*$ be prefix closed. For $(i,k) \in I \times K$, let

$$\varphi_i^{IK} : \Sigma_{IK}^* \to \Phi^* \text{ with } \varphi_i^{IK}(a_{rs}) = \begin{cases} a \mid a_{rs} \in \Phi_{\{i\}K} \\ \varepsilon \mid a_{rs} \in \Sigma_{IK} \setminus \Phi_{\{i\}K} \end{cases} \textit{and}$$

$$\gamma_k^{IK} : \Sigma_{IK}^* \to \Gamma^* \text{ with } \gamma_k^{IK}(a_{rs}) = \begin{cases} a \mid a_{rs} \in \Gamma_{I\{k\}} \\ \varepsilon \mid a_{rs} \in \Sigma_{IK} \setminus \Gamma_{I\{k\}} \end{cases},$$

where Φ_{IK} and Γ_{IK} are defined correspondingly to Σ_{IK}.

Definition 1. *Let I, K be finite parameter sets, then*

$$\mathcal{L}_{IK} = \bigcap_{(i,k)\in I\times K} (\pi_{ik}^{IK})^{-1}(L) \cap \bigcap_{i\in I} (\varphi_i^{IK})^{-1}(SF) \cap \bigcap_{k\in K} (\gamma_k^{IK})^{-1}(SG)$$

denotes a uniformly parameterised cooperation.

By this definition

$$\mathcal{L}_{\{1\}\{1\}} = (\pi_{11}^{\{1\}\{1\}})^{-1}(L) \cap (\varphi_1^{\{1\}\{1\}})^{-1}(SF) \cap (\gamma_1^{\{1\}\{1\}})^{-1}(SG).$$

As we want $\mathcal{L}_{\{1\}\{1\}}$ beeing isomorphic to L by the isomorphism $\pi_{11}^{\{1\}\{1\}} : \Sigma_{\{1\}\{1\}}^* \to \Sigma^*$ we additionally need $(\pi_{11}^{\{1\}\{1\}})^{-1}(L) \subset (\varphi_1^{\{1\}\{1\}})^{-1}(SF)$ and $(\pi_{11}^{\{1\}\{1\}})^{-1}(L) \subset (\gamma_1^{\{1\}\{1\}})^{-1}(SG)$. This is equivalent to $\pi_\Phi(L) \subset SF$ and $\pi_\Gamma(L) \subset SG$, where $\pi_\Phi : \Sigma^* \to \Phi^*$ and $\pi_\Gamma : \Sigma^* \to \Gamma^*$ are defined by

$$\pi_\Phi(a) = \begin{cases} a \mid a \in \Phi \\ \varepsilon \mid a \in \Gamma \end{cases} \text{ and } \pi_\Gamma(a) = \begin{cases} a \mid a \in \Gamma \\ \varepsilon \mid a \in \Phi \end{cases}.$$

So we complete definition 1 by the additional conditions $\pi_\Phi(L) \subset SF$ and $\pi_\Gamma(L) \subset SG$. Now we consider special abstractions on $\mathcal{L}_{IK}$.

Definition 2. *For $I' \subset I$ and $K' \subset K$ let*

$$\Pi_{I'K'}^{IK} : \Sigma_{IK}^* \to \Sigma_{I'K'}^* \textit{ with } \Pi_{I'K'}^{IK}(a_{rs}) = \begin{cases} a_{rs} \mid a_{rs} \in \Sigma_{I'K'} \\ \varepsilon \mid a_{rs} \in \Sigma_{IK} \setminus \Sigma_{I'K'} \end{cases}$$

Lemma 1. *For $I' \subset I$, $K' \subset K$, and, $L \subset \Sigma^*$, $SF \subset \Phi^*$, $SG \in \Gamma^*$ prefix closed* *and non-empty, the following relationships hold:*

$$(\varphi_i^{IK})^{-1}(\varepsilon) \supset \Sigma^*_{I'K} \textit{ for } i \in I \setminus I', \tag{1a}$$

$$(\gamma_k^{IK})^{-1}(\varepsilon) \supset \Sigma^*_{IK'} \textit{ for } k \in K \setminus K', \tag{1b}$$

$$\bigcap_{(i,k)\in(I\times K)\setminus(I'\times K')} [(\pi_{ik}^{IK})^{-1}(\varepsilon)] = \Sigma^*_{I'K'}, \tag{1c}$$

$$\Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)] = (\pi_{ik}^{I'K'})^{-1}(L) \textit{ for } (i,k) \in I \times K, \tag{1d}$$

$$\Pi^{IK}_{I'K'}[(\varphi_i^{IK})^{-1}(SF)] = \Sigma^*_{I'K'} \textit{ for } i \in I \setminus I', \tag{1e}$$

$$\Pi^{IK}_{I'K'}[(\gamma_k^{IK})^{-1}(SG)] = \Sigma^*_{I'K'} \textit{ for } k \in K \setminus K', \tag{1f}$$

$$\Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)] = \Sigma^*_{I'K'} \textit{ for } (i,k) \in (I \times K) \setminus (I' \times K'). \tag{1g}$$

Proof.

(1a) $\varphi_i^{IK}(x) = \varepsilon$ for each $x \in \Sigma^*_{I'K}$ and $i \in I \setminus I'$, which implies (1a).

(1b) Follows analogously.

(1c) For $x \in \Sigma^*_{IK}$, $x \in \Sigma^*_{I'K'}$ holds iff $\pi_{ik}^{IK}(x) = \varepsilon$ for each $(i,k) = (I\times K)\setminus(I'\times K')$, which implies (1c).

(1d) $x \in \Sigma^*_{I'K'}$ and $\pi_{ik}^{I'K'}(x) \in L$, for $x \in (\pi_{ik}^{I'K'})^{-1}(L)$. From this it follows that $x \in \Sigma^*_{IK}$, $\pi_{ik}^{IK}(x) = \pi_{ik}^{I'K'}(x) \in L$ and $x = \Pi^{IK}_{I'K'}(x)$, which implies $x \in \Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)]$. Hence $(\pi_{ik}^{I'K'})^{-1}(L) \subset \Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)]$. For $x \in \Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)]$ exists $y \in \Sigma^*_{IK}$ such that $\pi_{ik}^{IK}(y) \in L$ and $x = \Pi^{IK}_{I'K'}(y)$. Since $(i,k) \in I' \times K'$ it follows that $\pi_{ik}^{IK}(y) = \pi_{ik}^{I'K'}(\Pi^{IK}_{I'K'}(y)) = \pi_{ik}^{I'K'}(x) \in L$ which proves the inclusion $\Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)] \subset (\pi_{ik}^{I'K'})^{-1}(L)$.

(1e) For $x \in \Sigma^*_{I'K'}$ and $i \in I \setminus I'$ holds $x \in \Sigma^*_{IK}$, $\varphi_i^{IK}(x) = \varepsilon \in SF$ and $x \in \Pi^{IK}_{I'K'}(x)$, and so $x \in \Pi^{IK}_{I'K'}[(\varphi_i^{IK})^{-1}(SF)]$. Hence $\Sigma^*_{I'K'} \subset \Pi^{IK}_{I'K'}[(\varphi_i^{IK})^{-1}(SF)]$. The reverse inclusion holds because of $\Pi^{IK}_{I'K'} : \Sigma^*_{IK} \to \Sigma^*_{I'K'}$.

(1f) and (1g) The proofs are analogous to the proof for (1e).

□

Theorem 1.
$\mathcal{L}_{IK} \supset \mathcal{L}_{I'K'}$ *for* $I' \times K' \subset I \times K$, *and therefore*
$\Pi^{IK}_{I'K'}(\mathcal{L}_{IK}) \supset \Pi^{IK}_{I'K'}(\mathcal{L}_{I'K'}) = \mathcal{L}_{I'K'}$

Proof.
Because of (1a)-(1c)

$$
\begin{aligned}
\mathcal{L}_{IK} = & \bigcap_{(i,k)\in I\times K} [(\pi_{ik}^{IK})^{-1}(L) \cap (\varphi_i^{IK})^{-1}(SF) \cap (\gamma_k^{IK})^{-1}(SG)] \\
= & \bigcap_{(i,k)\in I'\times K'} [(\pi_{ik}^{IK})^{-1}(L) \cap (\varphi_i^{IK})^{-1}(SF) \cap (\gamma_k^{IK})^{-1}(SG)] \cap \\
& \cap \bigcap_{(i,k)\in (I\times K)\setminus(I'\times K')} (\pi_{ik}^{IK})^{-1}(L) \cap \bigcap_{i\in I\setminus I'} (\varphi_i^{IK})^{-1}(SF)) \cap \\
& \cap \bigcap_{k\in K\setminus K'} (\gamma_k^{IK})^{-1}(SG) \\
\supset & \bigcap_{(i,k)\in I'\times K'} [(\pi_{ik}^{IK})^{-1}(L) \cap (\varphi_i^{IK})^{-1}(SF) \cap (\gamma_k^{IK})^{-1}(SG)] \cap \\
& \cap \bigcap_{(i,k)\in (I\times K)\setminus(I'\times K')} (\pi_{ik}^{IK})^{-1}(\varepsilon) \cap \bigcap_{i\in I\setminus I'} (\varphi_i^{IK})^{-1}(\varepsilon)) \cap \bigcap_{k\in K\setminus K'} (\gamma_k^{IK})^{-1}(\varepsilon) \\
\supset & \bigcap_{(i,k)\in I'\times K'} [(\pi_{ik}^{IK})^{-1}(L) \cap (\varphi_i^{IK})^{-1}(SF) \cap (\gamma_k^{IK})^{-1}(SG)] \cap \Sigma_{I'K'}^* \\
= & \bigcap_{(i,k)\in I'\times K'} [(\pi_{ik}^{I'K'})^{-1}(L) \cap (\varphi_i^{I'K'})^{-1}(SF) \cap (\gamma_k^{I'K'})^{-1}(SG)] = \mathcal{L}_{I'K'}
\end{aligned}
$$

□

Examples show that the reverse inclusions

$$\Pi_{I'K'}^{IK}(\mathcal{L}_{IK}) \subset \mathcal{L}_{I'K'} \text{ for all } I' \times K' \subset I \times K \tag{2}$$

do not hold in general.

In the general case we don't know the decidability status of (2). But for many parameterised systems (2), and therefore $\Pi_{I'K'}^{IK}(\mathcal{L}_{IK}) = \mathcal{L}_{I'K'}$, which is a generalisation of $\pi_{ik}^{IK}(\mathcal{L}_{IK}) = L$, is a desirable property, because it describes a kind of self-similarity: From an abstracting point of view, where only the actions of $\Sigma_{I'K'}$ are considered, the complex system $\mathcal{L}_{IK}$ behaves like the smaller subsystem $\mathcal{L}_{I'K'}$. So we are looking for conditions, which imply (2). For this, some preliminary considerations are needed.

In general for a mapping $f : X \to Y$ and a family $(A_t)_{t\in T}$ of sets with $A_t \subset X$ for each $t \in T$:

$$f(\bigcap_{t\in T} A_t) \subset \bigcap_{t\in T} f(A_t) \tag{3}$$

The definition of $\mathcal{L}_{IK}$ can be transformed

$$\begin{aligned}\mathcal{L}_{IK} &= \bigcap_{(i,k)\in I\times K}(\pi_{ik}^{IK})^{-1}(L)\cap\bigcap_{i\in I}(\varphi_i^{IK})^{-1}(SF)\cap\bigcap_{k\in K}(\gamma_k^{IK})^{-1}(SG)\\
&= \bigcap_{(i,k)\in I'\times K'}(\pi_{ik}^{IK})^{-1}(L)\cap\bigcap_{i\in I'}(\varphi_i^{IK})^{-1}(SF)\cap\bigcap_{k\in K'}(\gamma_k^{IK})^{-1}(SG)\cap\\
&\qquad\bigcap_{(i,k)\in(I\times K)\setminus(I'\times K')}(\pi_{ik}^{IK})^{-1}(L)\cap\bigcap_{i\in I\setminus I'}(\varphi_i^{IK})^{-1}(SF)\cap\bigcap_{k\in K\setminus K'}(\gamma_k^{IK})^{-1}(SG)\\
&= \bigcap_{(i,k)\in I'\times K'}(\pi_{ik}^{IK})^{-1}(L)\cap\bigcap_{i\in I'}[[\bigcap_{s\in K}(\pi_{is}^{IK})^{-1}(L)]\cap(\varphi_i^{IK})^{-1}(SF)]\cap\\
&\qquad\bigcap_{k\in K'}[[\bigcap_{r\in I}(\pi_{rk}^{IK})^{-1}(L)]\cap(\gamma_k^{IK})^{-1}(SG)]\cap\\
&\qquad\bigcap_{(i,k)\in(I\times K)\setminus(I'\times K')}(\pi_{ik}^{IK})^{-1}(L)\cap\bigcap_{i\in I\setminus I'}(\varphi_i^{IK})^{-1}(SF)\cap\bigcap_{k\in K\setminus K'}(\gamma_k^{IK})^{-1}(SG)\end{aligned}$$

With (3) and (1d)-(1g) from this it follows that

$$\begin{aligned}\Pi_{I'K'}^{IK}(\mathcal{L}_{IK})\subset&\bigcap_{(i,k)\in I'\times K'}(\pi_{ik}^{I'K'})^{-1}(L)\cap\\
&\bigcap_{i\in I'}\Pi_{I'K'}^{IK}[[\bigcap_{s\in K}(\pi_{is}^{IK})^{-1}(L)]\cap(\varphi_i^{IK})^{-1}(SF)]\cap\\
&\bigcap_{k\in K'}\Pi_{I'K'}^{IK}[[\bigcap_{r\in I}(\pi_{rk}^{IK})^{-1}(L)]\cap(\gamma_k^{IK})^{-1}(SG)]\cap\Sigma_{I'K'}^{*}.\end{aligned}$$

Therefore, to prove (2), it is sufficient to show

$$\Pi_{I'K'}^{IK}[[\bigcap_{s\in K}(\pi_{is}^{IK})^{-1}(L)]\cap(\varphi_i^{IK})^{-1}(SF)]\subset(\varphi_i^{I'K'})^{-1}(SF)\tag{4}$$

for $I'\subset I$, $K'\subset K$, all $i\in I'$, and to show corresponding inclusions with respect to γ_k^{IK}, $\gamma_k^{I'K'}$ and SG for all $k\in K'$.
For $(r,s)\in I\times K$ let

$$\varphi_{rs}^{IK}:\Sigma_{IK}^{*}\to\Phi^{*}\text{ with }\varphi_{rs}^{IK}(a_{ik}):=\begin{cases}a\mid a_{ik}\in\Sigma_{IK},(r,s)=(i,k)\text{ and }a\in\Phi\\ \varepsilon\mid a_{ik}\in\Sigma_{IK},(r,s)\neq(i,k)\text{ or }a\notin\Phi\end{cases}$$

Hence $\varphi_{rs}^{IK}=\pi_\Phi\circ\pi_{rs}^{IK}$.
Because of $L\subset\pi_\Phi^{-1}(\pi_\Phi(L))$ and the precondition $\pi_\Phi(L)\subset SF$ it follows that $(\pi_{rs}^{IK})^{-1}(L)\subset(\pi_{rs}^{IK})^{-1}[\pi_\Phi^{-1}(\pi_\Phi(L))]=(\varphi_{rs}^{IK})^{-1}(\pi_\Phi(L))\subset(\varphi_{rs}^{IK})^{-1}(SF)$.
Therefore it is sufficient for the proof of (4) to show

$$\Pi_{I'K'}^{IK}[[\bigcap_{s\in K}(\varphi_{is}^{IK})^{-1}(SF)]\cap(\varphi_i^{IK})^{-1}(SF)]\subset(\varphi_i^{I'K'})^{-1}(SF).\tag{5}$$

Let

$$\pi_\Phi^{IK}:\Sigma_{IK}^{*}\to\Phi_{IK}^{*}\text{ with }\pi_\Phi^{IK}(a_{ik}):=\begin{cases}a_{ik}\mid a_{ik}\in\Phi_{IK}\\ \varepsilon\mid a_{ik}\in\Gamma_{IK}.\end{cases}$$

Let

$$\bar\varphi_i^{IK}:\Phi_{IK}^{*}\to\Phi^{*}\text{ with }\bar\varphi_i^{IK}(x):=\varphi_i^{IK}(x)$$

and

$$\bar{\varphi}_{ik}^{IK} : \Phi_{IK}^* \to \Phi^* \text{ with } \bar{\varphi}_{ik}^{IK}(x) := \varphi_{ik}^{IK}(x)$$

for each $x \in \Phi_{IK}^*$ and $(i,k) \in I \times K$.
So

$$\varphi_i^{IK} = \bar{\varphi}_i^{IK} \circ \pi_\Phi^{IK} \text{ and } \varphi_{ik}^{IK} = \bar{\varphi}_{ik}^{IK} \circ \pi_\Phi^{IK} \text{ for } (i,k) \in I \times K, \tag{6}$$

as well as

$$\Pi_{I'K'}^{IK}((\pi_\Phi^{IK})^{-1}(y)) = (\pi_\Phi^{I'K'})^{-1}(\Pi_{I'K'}^{IK}(y)) \text{ for each } y \in \Phi_{IK}^*, \tag{7}$$

where $\pi_\Phi^{I'K'} : \Sigma_{I'K'}^* \to \Phi_{I'K'}^*$ is defined corresponding to π_Φ^{IK}.

(7) is a special case of Lemma 2, which will be proven below.

Because of (6) and (7) now we have
$\Pi_{I'K'}^{IK}[[\bigcap_{s\in K}(\varphi_{is}^{IK})^{-1}(SF)] \cap (\varphi_i^{IK})^{-1}(SF)] =$
$= \Pi_{I'K'}^{IK}[(\pi_\Phi^{IK})^{-1}[[\bigcap_{s\in K}(\bar{\varphi}_{is}^{IK})^{-1}(SF)] \cap (\bar{\varphi}_i^{IK})^{-1}(SF)]] =$
$= (\pi_\Phi^{I'K'})^{-1}[\Pi_{I'K'}^{IK}[[\bigcap_{s\in K}(\bar{\varphi}_{is}^{IK})^{-1}(SF)] \cap (\bar{\varphi}_i^{IK})^{-1}(SF)]],$
and $(\varphi_i^{I'K'})^{-1}(SF) = (\pi_\Phi^{I'K'})^{-1}[(\bar{\varphi}_i^{I'K'})^{-1}(SF)]$.
For the proof of (5) it is therefore sufficient to show

$$\Pi_{I'K'}^{IK}[[\bigcap_{s\in K}(\bar{\varphi}_{is}^{IK})^{-1}(SF)] \cap (\bar{\varphi}_i^{IK})^{-1}(SF)] \subset (\bar{\varphi}_i^{I'K'})^{-1}(SF). \tag{8}$$

For index sets S and T and $S' \times T' \subset S \times T$ let

$$\hat{\varphi}_{S'T'}^{ST} : \Phi_{ST}^* \to \Phi_{S'T'}^* \text{ with } \hat{\varphi}_{S'T'}^{ST}(x) := \Pi_{S'T'}^{ST}(x) \text{ for } x \in \Phi_{ST}^* \subset \Sigma_{ST}^*.$$

With this (8) is equivalent to

$$\hat{\varphi}_{I'K'}^{IK}[[\bigcap_{s\in K}(\bar{\varphi}_{is}^{IK})^{-1}(SF)] \cap (\bar{\varphi}_i^{IK})^{-1}(SF)] \subset (\bar{\varphi}_i^{I'K'})^{-1}(SF). \tag{9}$$

From the definitions it follows that

$$\bar{\varphi}_{ik}^{IK} = \bar{\varphi}_{ik}^{\{i\}K} \circ \hat{\varphi}_{\{i\}K}^{IK} \text{ and } \bar{\varphi}_i^{IK} = \bar{\varphi}_i^{\{i\}K} \circ \hat{\varphi}_{\{i\}K}^{IK} \text{ for } (i,k) \in I \times K. \tag{10}$$

For $I' \times K' \subset I \times K$ and $i \in I'$ holds

$$\hat{\varphi}_{I'K'}^{IK}((\hat{\varphi}_{\{i\}K}^{IK})^{-1}(y)) = (\hat{\varphi}_{\{i\}K'}^{I'K})^{-1}(\hat{\varphi}_{I'K'}^{IK}(y)) \text{ for } y \in \Phi_{\{i\}K}^*. \tag{11}$$

(11) is a special case of Lemma 2, which will be proven below.

Because of (10) and (11) it holds
$\hat{\varphi}_{I'K'}^{IK}[[\bigcap_{s\in K}(\bar{\varphi}_{is}^{IK})^{-1}(SF)] \cap (\bar{\varphi}_i^{IK})^{-1}(SF)] =$
$= (\hat{\varphi}_{\{i\}K'}^{I'K'})^{-1}[\hat{\varphi}_{I'K'}^{IK}[[\bigcap_{s\in K}(\bar{\varphi}_{is}^{\{i\}K})^{-1}(SF)] \cap (\bar{\varphi}_i^{\{i\}K})^{-1}(SF)]$ and
$(\bar{\varphi}_i^{I'K'})^{-1}(SF) = (\hat{\varphi}_{\{i\}K'}^{I'K'})^{-1}[(\bar{\varphi}_i^{\{i\}K})^{-1}(SF)]$ for $i \in I'$.
So for the proof of (9) it is sufficient to show

$$\hat{\varphi}_{I'K'}^{IK}[[\bigcap_{s\in K}(\bar{\varphi}_{is}^{\{i\}K})^{-1}(SF)] \cap (\bar{\varphi}_i^{\{i\}K})^{-1}(SF)] \subset (\bar{\varphi}_i^{\{i\}K'})^{-1}(SF) \tag{12}$$

Since $\hat{\varphi}^{IK}_{I'K'}(x) = \hat{\varphi}^{\{i\}K}_{\{i\}K'}(x)$ for $x \in \Phi^*_{\{i\}K}$ and $i \in I'$, (12) is equivalent to

$$\hat{\varphi}^{\{i\}K}_{\{i\}K'}[[\bigcap_{s\in K}(\bar{\varphi}^{\{i\}K}_{is})^{-1}(SF)] \cap (\bar{\varphi}^{\{i\}K}_{i})^{-1}(SF)] \subset (\bar{\varphi}^{\{i\}K'}_{i})^{-1}(SF) \qquad (13)$$

(13) has to be proven for arbitray index sets $K' \subset K$ and each $i \in I' \subset I$.
From the definitions of $\hat{\varphi}^{\{i\}K}_{\{i\}K'}$, $\bar{\varphi}^{\{i\}K}_{is}$, $\bar{\varphi}^{\{i\}K}_{i}$, and $\bar{\varphi}^{\{i\}K'}_{i}$ it follows directly that for each $i \in I'$, (13) is the same inclusion

$$\hat{\varphi}^{K}_{K'}[[\bigcap_{s\in K}(\bar{\varphi}^{K}_{s})^{-1}(SF)] \cap (\bar{\varphi}^{K})^{-1}(SF)] \subset (\bar{\varphi}^{K'})^{-1}(SF) \qquad (14)$$

up to isomorphism.
For definition of the homomorphisms occurring therein, let K be an arbitrary index set, and for each $s \in K$ let Φ_s be a copy of Φ. Let all Φ_s be pairwise disjoint. The index s describes the bijection $a \leftrightarrow a_s$ for $a \in \Phi$ and $a_s \in \Phi_s$, and $\Phi_K := \dot{\bigcup}_{s\in K} \Phi_s$.

For $K' \subset K$, let $\hat{\varphi}^{K}_{K'} : \Phi^*_K \to \Phi^*_{K'}$ with $\hat{\varphi}^{K}_{K'}(a_r) := \begin{cases} a_r \mid a_r \in \Phi_{K'} \\ \varepsilon \mid a_r \in \Phi_K \setminus \Phi_{K'} \end{cases}$.

For $s \in K$, let $\bar{\varphi}^{K}_{s} : \Phi^*_K \to \Phi^*$ with $\bar{\varphi}^{K}_{s}(a_r) := \begin{cases} a \mid a_r \in \Phi_s \\ \varepsilon \mid a_r \in \Phi_K \setminus \Phi_s \end{cases}$.

Let $\bar{\varphi}^{K} : \Phi^*_K \to \Phi^*$ with $\bar{\varphi}^{K}(a_r) := a$ for each $r \in K$ and $a_r \in \Phi_r$.

Now, up to equations (7) and (11), we have shown the following

Theorem 2.
Let $I' \times K' \subset I \times K$. Assuming (14) and a corresponding inclusion concerning SG, then $\Pi^{IK}_{I'K'}(\mathcal{L}_{IK}) = \mathcal{L}_{I'K'}$.

To prove (14) we have to show that

$$\hat{\varphi}^{K}_{K'}(w) \in (\bar{\varphi}^{K'})^{-1}(SF) \text{ for each } w \in [\bigcap_{s\in K}(\bar{\varphi}^{K}_{s})^{-1}(SF)] \cap (\bar{\varphi}^{K})^{-1}(SF). \qquad (15)$$

Equations (7) and (11) are special cases of a more general lemma.
For a set X and $X' \subset X$ let

$$\pi^X_{X'} : X^* \to X'^* \text{ with } \pi^X_{X'}(a) := \begin{cases} a \mid a \in X' \\ \varepsilon \mid a \in X \setminus X' \end{cases}$$

Lemma 2 (projection-lemma)**.**
Let $X' \subset X$, $Y \subset X$ and $Y' := X' \cap Y$, then

$$\pi^X_{X'}((\pi^X_Y)^{-1}(y)) = (\pi^{X'}_{Y'})^{-1}(\pi^X_{X'}(y)) \textit{ for each } y \in Y^* \qquad (16)$$

Proof. Let $y \in Y^*$. We show

$$\pi^{X'}_{Y'}(\pi^X_{X'}(z)) = \pi^X_{X'}(y) \text{ for each } z \in (\pi^X_Y)^{-1}(y) \qquad (17)$$

and we show that

$$\text{for each } u \in (\pi^{X'}_{Y'})^{-1}(\pi^X_{X'}(y)) \text{ there exists a } v \in (\pi^X_Y)^{-1}(y) \text{ such that } \pi^X_{X'}(v) = u. \qquad (18)$$

From (17) it follows that $\pi^X_{X'}((\pi^X_Y)^{-1}(y)) \subset (\pi^{X'}_{Y'})^{-1}(\pi^X_{X'}(y))$ and from (18) it follows that $(\pi^{X'}_{Y'})^{-1}(\pi^X_{X'}(y)) \subset \pi^X_{X'}((\pi^X_Y)^{-1}(y))$ which in turn proves (16).

Proof of (17):
By definition of π_Y^X, $\pi_{Y'}^{X'}$ and $\pi_{X'}^X$ follows $\pi_{Y'}^{X'}(\pi_{X'}^X(z)) = \pi_{X'}^X(\pi_Y^X(z))$ for each $z \in X^*$ and therewith (17).
Proof of (18) by induction on $y \in Y^*$:
Induction base.
Let $y = \varepsilon$, then $u \in (X' \setminus Y')^*$ for each $u \in (\pi_{Y'}^{X'})^{-1}(\pi_{X'}^X(y))$.
From this follows $\pi_{X'}^X(v) = u$ with $v := u \in (\pi_Y^X)^{-1}(\varepsilon)$.
Induction step. Let $y = \mathring{y}\hat{y}$ with $\mathring{y} \in Y^*$ and $\hat{y} \in Y$.
Case 1: $\hat{y} \in Y \setminus Y' = Y \cap (X \setminus X')$
Then $(\pi_{Y'}^{X'})^{-1}(\pi_{X'}^X(y)) = (\pi_{Y'}^{X'})^{-1}(\pi_{X'}^X(\mathring{y}))$.
By induction hypothesis then for each $u \in (\pi_{Y'}^{X'})^{-1}(\pi_{X'}^X(y))$ it exists $\mathring{v} \in (\pi_Y^X)^{-1}(\mathring{y})$ such that $\pi_{X'}^X(\mathring{v}) = u$.
With $v := \mathring{v}\hat{y}$ holds $\pi_Y^X(\mathring{v}\hat{y}) = \mathring{y}\hat{y} = y$ and hence $v \in (\pi_Y^X)^{-1}(y)$ and $\pi_{X'}^X(v) = \pi_{X'}^X(\mathring{v}) = u$.
Case 2: $\hat{y} \in Y' \subset X'$
Then $\pi_{X'}^X(y) = \pi_{X'}^X(\mathring{y})\hat{y}$. Therefore each $u \in (\pi_{Y'}^{X'})^{-1}(\pi_{X'}^X(y))$ can be departed into $u = \mathring{u}\hat{y}\hat{u}$ with $\mathring{u} \in (\pi_{Y'}^{X'})^{-1}(\pi_{X'}^X(\mathring{y}))$ and $\hat{u} \in (X' \setminus Y')^*$. By induction hypothesis then exists $\mathring{v} \in (\pi_Y^X)^{-1}(\mathring{y})$ such that $\pi_{X'}^X(\mathring{v}) = \mathring{u}$.
With $v := \mathring{v}\hat{y}\hat{u}$ holds $\pi_Y^X(\mathring{v}\hat{y}\hat{u}) = \mathring{y}\hat{y} = y$ and hence $v \in (\pi_Y^X)^{-1}(y)$ and $\pi_{X'}^X(v) = \pi_{X'}^X(\mathring{v})\hat{y}\hat{u} = \mathring{u}\hat{y}\hat{u} = u$.
This completes the proof of (18). □

Remark. (7) follows from (16) by $X = \Sigma_{IK}$, $X' = \Sigma_{I'K'}$ and $Y = \Phi_{IK}$. (11) follows from (16) by $X = \Phi_{IK}$, $X' = \Phi_{I'K'}$ and $Y = \Phi_{\{i\}K}$.

3. SCHEDULES BASED ON PHASES

By definitions of $\bar{\varphi}_s^K$ and $\bar{\varphi}^K$ it holds

$$\bar{\varphi}^K(w) \in SF^{\sqcup\!\sqcup} \text{ for each } w \in [\bigcap_{s \in K} (\bar{\varphi}_s^K)^{-1}(SF)] \cap (\bar{\varphi}^K)^{-1}(SF),$$

where $SF^{\sqcup\!\sqcup}$ denotes the *iterated shuffle product* of SF.

Definition 3.

$$P^{\sqcup\!\sqcup} := \Theta^{\mathbb{N}}[\bigcap_{t \in \mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})] \textit{ for } P \subset \Sigma^*.$$

For the definition of the homomorphisms $\Theta^{\mathbb{N}}$ and $\tau_t^{\mathbb{N}}$, let $t \in \mathbb{N}$, and for each t let Σ_t be a copy of Σ. Let all Σ_t be pairwise disjoint. The index t describes the bijection $a \leftrightarrow a_t$ for $a \in \Sigma$ and $a_t \in \Sigma_t$.

Let $\Sigma_{\mathbb{N}} := \bigcup_{t \in \mathbb{N}} \Sigma_t$, and for each $t \in \mathbb{N}$ let the homomorphisms $\tau_t^{\mathbb{N}}$ and $\Theta^{\mathbb{N}}$ be defined by

$$\tau_t^{\mathbb{N}} : \Sigma_{\mathbb{N}}^* \to \Sigma^* \text{ with } \tau_t^{\mathbb{N}}(a_s) = \begin{cases} a \mid a_s \in \Sigma_t \\ \varepsilon \mid a_s \in \Sigma_{\mathbb{N}} \setminus \Sigma_t \end{cases}$$

and $\Theta^{\mathbb{N}} : \Sigma_{\mathbb{N}}^* \to \Sigma^*$ with $\Theta^{\mathbb{N}}(a_t) := a$ for $a_t \in \Sigma_t$ and $t \in \mathbb{N}$.

If we similarly define $\tau_k^K : \Sigma_K^* \to \Sigma^*$ for $k \in K$ and $\Theta^K : \Sigma_K^* \to \Sigma^*$ for arbitrary index sets K, then we have $\bar{\varphi}_k^K = \tau_k^K$ and $\bar{\varphi}^K = \Theta^K$ for $\Phi = \Sigma$.

Definition 3 looks different to the usual one of iterated shuffle products, as for example in [Jantzen 1985]. But it is easy to see that they are equivalent. We use our kind of definition, as it is more adequate to the considerations in this paper.

Directly from the definition it follows that $\text{pre}(P^{\sqcup\!\sqcup}) = (\text{pre}(P))^{\sqcup\!\sqcup}$ and $\tilde{P}^{\sqcup\!\sqcup} \subset P^{\sqcup\!\sqcup}$ for $\tilde{P} \subset P$. A detailed analysis of the structure of iterated shuffle products will pave the way for a sufficient condition for (14). For that, additional definitions and lemmata are required.

Definition 4.
$SF \subset \Phi^*$ *is based on a phase* $PF \subset \Phi^*$*, iff* $SF = \text{pre}(PF^{\sqcup\!\sqcup} \cap SF)$.

If SF is based on PF, then $SF \subset \text{pre}(PF^{\sqcup\!\sqcup}) = (\text{pre}(PF))^{\sqcup\!\sqcup}$ and $SF = \text{pre}(PF))^{\sqcup\!\sqcup} \cap SF$.
Furthermore, it follows that $SF^{\sqcup\!\sqcup} \subset ((\text{pre}(PF))^{\sqcup\!\sqcup})^{\sqcup\!\sqcup}$.

For the subsequent considerations let S and T be arbitrary index sets and $M \subset \Sigma^*$. For each $S' \subset S$ and $T' \subset T$ let

$$\Theta_{S'}^{S'\times T'} : \Sigma_{S'\times T'}^* \to \Sigma_{S'}^* \text{ with } \Theta_{S'}^{S'\times T'}(a_{(s,t)}) := a_s \text{ for each } a_{(s,t)} \in \Sigma_{S'\times T'} \text{ and}$$

$$\Theta_{T'}^{S'\times T'} : \Sigma_{S'\times T'}^* \to \Sigma_{T'}^* \text{ with } \Theta_{T'}^{S'\times T'}(a_{(s,t)}) := a_t \text{ for each } a_{(s,t)} \in \Sigma_{S'\times T'}.$$

Lemma 3 (Shuffle-lemma 1)**.**
Let S, T arbitrary index sets and $M \subset \Sigma^*$, then

$$\bigcap_{s\in S}(\tau_s^S)^{-1}[\Theta^T(\bigcap_{t\in T}(\tau_t^T)^{-1}(M))] = \Theta_S^{S\times T}[\bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(M)], \tag{19a}$$

and, since $\Theta^{S\times T} = \Theta^S \circ \Theta_S^{S\times T}$,

$$\Theta^S[\bigcap_{s\in S}(\tau_s^S)^{-1}[\Theta^T(\bigcap_{t\in T}(\tau_t^T)^{-1}(M))]] = \Theta^{S\times T}[\bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(M)]. \tag{19b}$$

Proof. For $x \in \Sigma_S^*$ let
$U_x := \{(y_s)_{s\in S} \in [\bigcap_{t\in T}(\tau_t^T)^{-1}(M)]^S \mid \tau_s^S(x) = \Theta^T(y_s) \text{ for each } s \in S\}$ and
$V_x := \{z \in \bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(M) \mid \Theta_S^{S\times T}(z) = x\}$.
Then $x \in \bigcap_{s\in S}(\tau_s^S)^{-1}[\Theta^T(\bigcap_{t\in T}(\tau_t^T)^{-1}(M))] \Leftrightarrow U_x \neq \emptyset$ and
$x \in \Theta_S^{S\times T}[\bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(M)] \Leftrightarrow V_x \neq \emptyset$.
Hence $\bigcap_{s\in S}(\tau_s^S)^{-1}[\Theta^T(\bigcap_{t\in T}(\tau_t^T)^{-1}(M))] = \Theta_S^{S\times T}[\bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(M)]$
iff $U_x \neq \emptyset \Leftrightarrow V_x \neq \emptyset$ for each $x \in \Sigma_S^*$.

These equivalences hold, if for each $x \in \Sigma_S^*$ a surjective mapping $\kappa_x : V_x \to U_x$ exists.
For $x \in \Sigma_S^*$ and $z \in \bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(M)$ let therefore $\kappa_x(z) := (y_s)_{s\in S}$ with $y_s = \Theta_T^{\{s\}\times T}(\Pi_{\{s\}T}^{ST}(z))$.

From this it follows that $\tau_t^T(y_s) = \tau_t^T(\Theta_T^{\{s\}\times T}(\Pi_{\{s\}T}^{ST}(z))) = \tau_{(s,t)}^{S\times T}(z) \in M$ for each $(s,t) \in S \times T$, thus $y_s \in \bigcap_{t\in T}(\tau_t^T)^{-1}(M)$ for each $s \in S$.

If $x = \Theta_S^{S\times T}(z)$, then $\tau_s^S(x) = \Theta^{\{s\}\times T}(\Pi_{\{s\}T}^{ST}(z)) = \Theta^T(\Theta_T^{\{s\}\times T}(\Pi_{\{s\}T}^{ST}(z))) = \Theta^T(y_s)$ for each $s \in S$ and hence $(y_s)_{s\in S} \in U_x$.
Therefore, κ_x defines a mapping $\kappa_x : V_x \to U_x$.
κ_x is surjective:
For each $s \in S$ and $y_s \in \bigcap_{t\in T}(\tau_t^T)^{-1}(M)$ exists an $y_s' \in \bigcap_{t\in T}(\tau_{(s,t)}^{\{s\}\times T})^{-1}(M)$ such that $y_s = \Theta_T^{\{s\}\times T}(y_s')$. If additionally $\tau_s^S(x) = \Theta^T(y_s) = \Theta^T(\Theta_T^{\{s\}\times T}(y_s')) = \Theta^{\{s\}\times T}(y_s')$ for each $s \in S$, then there exists $z \in \bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(M)$ with $y_s' = \Pi_{\{s\}T}^{ST}(z)$ and $\Theta_S^{S\times T}(z) = x$. For this z now $\kappa_x(z) = (\Theta_T^{\{s\}\times T}(\Pi_{\{s\}T}^{ST}(z)))_{s\in S} = (\Theta_T^{\{s\}\times T}(y_s'))_{s\in S} = (y_s)_{s\in S}$ and hence κ_x is surjective. □

Definition 5. *Let S be an arbitrary index set. For each $x \in \Theta^S[\bigcap_{s\in S}(\tau_s^S)^{-1}(M)]$ there exists $u \in \bigcap_{s\in S}(\tau_s^S)^{-1}(M)$ such that $x = \Theta^S(u)$. We call u a* structured representation of x w.r.t. S. *For $x \in \Sigma^*$ let $SR_M^S(x) := (\Theta^S)^{-1}(x) \cap [\bigcap_{s\in S}(\tau_s^S)^{-1}(M)]$. It is the set of all structured representations of x w.r.t. S and fixed $M \subset \Sigma^*$.*

Now $x \in P^{\sqcup\!\sqcup}$ iff there exists a countable index set S with $SR_{(P\cup\{\epsilon\})}^S(x) \neq \emptyset$ (see Lemma 4). If $x \in P^{\sqcup\!\sqcup}$, then generally $SR_{(P\cup\{\epsilon\})}^S(x)$ contains more than one element.

If $\iota : S \to T$ is a bijection, then it defines an isomorphism $\nu_\iota : \Sigma_S^* \to \Sigma_T^*$ with $\nu_\iota(a_s) = a_{\iota(s)}$ for each $s \in S$. For this isomorphism holds
$\nu_\iota(SR_M^S(x)) = \nu_\iota[(\Theta^S)^{-1}(x)] \cap \nu_\iota[\bigcap_{s\in S}(\tau_s^S)^{-1}(M)] = (\Theta^T)^{-1}(x) \cap [\bigcap_{s\in S}\nu_\iota((\tau_s^S)^{-1}(M))]$.
Since $\tau_{\iota(s)}^T \circ \nu_\iota = \tau_s^S$, it follows that
$\nu_\iota(SR_M^S(x)) = (\Theta^T)^{-1}(x) \cap [\bigcap_{s\in S}(\tau_{\iota(s)}^T)^{-1}(M)] = (\Theta^T)^{-1}(x) \cap [\bigcap_{t\in T}(\tau_t^T)^{-1}(M)] = SR_M^T(x)$. In summary we have

Lemma 4 (Shuffle-lemma 2)**.**
If a bijection between S and T exists, then $\Theta^S[\bigcap_{s\in S}(\tau_s^S)^{-1}(M)] = \Theta^T[\bigcap_{t\in T}(\tau_t^T)^{-1}(M)]$ for $M \subset \Sigma^*$.

For an arbitrary index set S and $S' \subset S$ let

$$\Pi_{S'}^S : \Sigma_S^* \to \Sigma_{S'}^* \quad \text{with } \Pi_{S'}^S(a_s) = \begin{cases} a_s \mid a_s \in \Sigma_{S'} \\ \varepsilon \mid a_s \in \Sigma_S \setminus \Sigma_{S'} \end{cases}.$$

Lemma 5 (Shuffle-lemma 3)**.**
Let $M \subset \Sigma^*$, S, T index sets and $y \in \Sigma_{S\times T}^*$ with $\tau_{(s,t)}^{S\times T}(y) \in M$ for each $(s,t) \in S \times T$ and $x = \Theta_S^{S\times T}(y) \in \Sigma_S^*$, then $\Pi_{S'\times T}^{S\times T}(y) \in SR_M^{S'\times T}(\Theta^{S'}(\Pi_{S'}^S(x)))$ for each $S' \subset S$.

Remark. The hypotheses of this lemma are given by (19a).

Proof. It holds $\Pi^{S\times T}_{S'\times T}(y) \in \Sigma^*_{S'\times T}$.

For $a_{(s,t)} \in \Sigma_{S\times T} \setminus \Sigma_{S'\times T}$ holds $\Theta^{S'\times T}(\Pi^{S\times T}_{S'\times T}(a_{(s,t)})) = \varepsilon$ and $\Theta^{S'}(\Pi^{S}_{S'}(\Theta^{S\times T}_{S}(a_{(s,t)}))) = \Theta^{S'}(\Pi^{S}_{S'}(a_s)) = \varepsilon$.
For $a_{(s,t)} \in \Sigma_{S'\times T}$ holds $\Theta^{S'\times T}(\Pi^{S\times T}_{S'\times T}(a_{(s,t)})) = \Theta^{S'\times T}(a_{(s,t)}) = a$ and $\Theta^{S'}(\Pi^{S}_{S'}(\Theta^{S\times T}_{S}(a_{(s,t)}))) = \Theta^{S'}(\Pi^{S}_{S'}(a_s)) = a$.
This implies $\Theta^{S'\times T}(\Pi^{S\times T}_{S'\times T}(y)) = \Theta^{S'}(\Pi^{S}_{S'}(x))$.
For $(s,t) \in S' \times T$ holds $\tau^{S'\times T}_{(s,t)}(\Pi^{S\times T}_{S'\times T}(y)) = \tau^{S\times T}_{(s,t)}(y) \in M$. This proves Lemma 5. □

We will now apply the three shuffle lemmata to the expressions of (14) and (15). To do this, the terminology has to be adapted. With the substitution $\Phi = \Sigma$, it holds $\hat{\varphi}^K_{K'} = \Pi^K_{K'}$, $\bar{\varphi}^K_k = \tau^K_k$ and $\bar{\varphi}^K = \Theta^K$.
If SF is based on PF, then $SF \subset (\text{pre}(PF))^{\sqcup\!\sqcup}$, thus by Lemma 3

$$\begin{aligned}\bigcap_{s\in K}(\bar{\varphi}^K_s)^{-1}(SF) &= \bigcap_{s\in K}(\tau^K_s)^{-1}(SF) \subset \bigcap_{s\in K}(\tau^K_s)^{-1}[(\text{pre}(PF))^{\sqcup\!\sqcup}] \\ &= \bigcap_{s\in K}(\tau^K_s)^{-1}[\Theta^{\mathbb{N}}(\bigcap_{t\in\mathbb{N}}(\tau^{\mathbb{N}}_t)^{-1}(\text{pre}(PF)))] \\ &= \Theta^{K\times\mathbb{N}}_K[\bigcap_{(s,t)\in(K\times\mathbb{N})}(\tau^{K\times\mathbb{N}}_{(s,t)})^{-1}(\text{pre}(PF))].\end{aligned}$$

For $w \in \bigcap_{s\in K}(\bar{\varphi}^K_s)^{-1}(SF)$ this implies $w = \Theta^{K\times\mathbb{N}}_K(y)$, and $\bar{\varphi}^K(w) = \Theta^K(\Theta^{K\times\mathbb{N}}_K(y)) = \Theta^{K\times\mathbb{N}}(y)$ for an $y \in \bigcap_{(s,t)\in(K\times\mathbb{N})}(\tau^{K\times\mathbb{N}}_{(s,t)})^{-1}(\text{pre}(PF))$.
Hence

$$y \in SR^{K\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^K(w)). \tag{20}$$

According to Lemma 5 for $K' \subset K$ now:

$$\Pi^{K\times\mathbb{N}}_{K'\times\mathbb{N}}(y) \in SR^{K'\times\mathbb{N}}_{\text{pre}(PF)}(\Theta^{K'}(\Pi^K_{K'}(\Theta^{K\times\mathbb{N}}_K(y)))) = SR^{K'\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w))). \tag{21}$$

Generally for $u \in SF$ an $y \in SR^{K\times\mathbb{N}}_{\text{pre}(PF)}(u)$ describes "how the cooperation partner F is involved in several phases".
Unfortunately this is ambiguous if $SR^{K\times\mathbb{N}}_{\text{pre}(PF)}(u)$ contains more than one element.

If $K' \neq \emptyset$ and K is finite, then $K \times \mathbb{N}$ and $K' \times \mathbb{N}$ are countable, thus isomorphic to $\mathbb{N}$.
From the existence of $y \in SR^{K\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^K(w))$ and likewise of $\Pi^{K\times\mathbb{N}}_{K'\times\mathbb{N}}(y) \in SR^{K'\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w))$, it follows by Lemma 4 that $\bar{\varphi}^K(w) \in (\text{pre}(PF))^{\sqcup\!\sqcup}$ as well as $\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w)) \in (\text{pre}(PF))^{\sqcup\!\sqcup}$.
Later we will use these considerations for a sufficient condition for (14).

The idea of such a condition is the following: For a cooperation partner the "possibilities of acting in a phase" depend on a "kind of resources". So the "more phases a partner is involved in", the less possibilities of acting in each phase he has.

To formalise this intuition we need an unambiguous description of "how a cooperation partner is involved in several phases". This will be done by an automaton representation of the iterated shuffle product.

4. SHUFFLE AUTOMATA

Let $P \subset \Sigma^*$ and $\mathbb{A} = (\Sigma, Q, \Delta, q_0, F)$ with $\Delta \subset Q \times \Sigma \times Q$, $q_0 \in Q$ and $F \subset Q$ be an (not necessarily finite) automaton that accepts P. To exclude pathological cases we assume $\varepsilon \notin P \neq \emptyset$. A consequence of this is in particular that $q_0 \notin F$.

For the construction of $\mathbb{A}^{\sqcup\!\sqcup}$ the set $\mathbb{N}_0^Q$ (set of all functions from Q in $\mathbb{N}_0$) plays a central role. In $\mathbb{N}_0^Q$ we distinguish the following functions:
$0 \in \mathbb{N}_0^Q$ with $0(x) = 0$ for each $x \in Q$, and for $q \in Q$ the function $1_q \in \mathbb{N}_0^Q$ with
$1_q(x) = \begin{cases} 1 \mid x = q \\ 0 \mid x \in Q \setminus \{q\} \end{cases}$.

As usual for numerical functions, a partial order as well as addition and partial subtraction are defined:

For $f, g \in \mathbb{N}_0^Q$ let

- $f \geqslant g$ iff $f(x) \geqslant g(x)$ for each $x \in Q$,
- $f + g \in \mathbb{N}_0^Q$ with $(f + g)(x) := f(x) + g(x)$ for each $x \in Q$, and
- for $f \geqslant g$, $f - g \in \mathbb{N}_0^Q$ with $(f - g)(x) := f(x) - g(x)$ for each $x \in Q$.

The key idea of $\mathbb{A}^{\sqcup\!\sqcup}$ is, to record in the functions of $\mathbb{N}_0^Q$ how many "open phases" are in each state $q \in Q$ respectively. Its state transition relation $\Delta^{\sqcup\!\sqcup}$ is composed of four subsets whose elements describe

- the "entry into a new phase",
- the "transition within an open phase",
- the "completion of an open phase",
- the "entry into a new phase with simultaneous completion of this phase".

With these definitions we now define the *shuffle automaton* $\mathbb{A}^{\sqcup\!\sqcup}$ as follows:

Definition 6 (shuffle automaton)**.**
The shuffle automaton $\mathbb{A}^{\sqcup\!\sqcup} = (\Sigma, \mathbb{N}_0^Q, \Delta^{\sqcup\!\sqcup}, 0, \{0\})$ *w.r.t.* $\mathbb{A}$ *is an automaton with infinite state set* $\mathbb{N}_0^Q$ *and*

$$\begin{aligned}\Delta^{\sqcup\!\sqcup} :=& \{(f, a, f + 1_p) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid (q_0, a, p) \in \Delta \textit{ and it exists } (p, x, y) \in \Delta\} \cup \\ &\cup\ \{(f, a, f + 1_p - 1_q) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid f \geqslant 1_q, (q, a, p) \in \Delta \textit{ and it exists} \\ &(p, x, y) \in \Delta\} \cup \\ &\cup\ \{(f, a, f - 1_q) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid f \geqslant 1_q, (q, a, p) \in \Delta \textit{ and } p \in F\} \cup \\ &\cup\ \{(f, a, f) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid (q_0, a, p) \in \Delta \textit{ and } p \in F\}.\end{aligned}$$

Generally $\mathbb{A}^{\sqcup\!\sqcup}$ is a non-deterministic automaton. In the literature such automata are called multicounter automata [Björklund and Bojanczyk 2007] and it is known that they accept the iterated shuffle products [Jedrzejowicz 1999]. For our purposes deterministic computations of these automata are very important. To analyse these aspects more deeply we use our own notation and proof of the main theorems.

To prove the following theorem for simplicity we assume that $\mathbb{A}$ is deterministic. I.e., the state transition relation Δ can be described by a partial function δ :

$Q \times \Sigma \to Q$ which is extended to a partial function $\delta : Q \times \Sigma^* \to Q$ as usual [Sakarovitch 2009]. Additionally we assume that $\mathbb{A}$ does not contain superfluous states, i.e. $\delta(q_0, \mathrm{pre}(P)) = Q$. So $\Delta^{\sqcup}$ can be represented by

$$
\begin{aligned}
\Delta^{\sqcup} =& \tilde{\Delta} \cup \mathring{\Delta} \cup \bar{\Delta} \cup \tilde{\tilde{\Delta}} \text{ with} \\
\tilde{\Delta} =& \{(f, a, f + 1_p) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q_0, a) = p \text{ and it exists } b \in \Sigma \text{ such that } \\
& \delta(p, b) \text{ is defined}\}, \\
\mathring{\Delta} =& \{(f, a, f + 1_p - 1_q) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid f \geqslant 1_q, \delta(q, a) = p \text{ and it exists } b \in \Sigma \\
& \text{such that } \delta(p, b) \text{ is defined}\}, \\
\bar{\Delta} =& \{(f, a, f - 1_p) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid f \geqslant 1_q \text{ and } \delta(q, a) \in F\} \text{ and} \\
\tilde{\tilde{\Delta}} =& \{(f, a, f) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q_0, a) \in F\}.
\end{aligned}
$$

Let $A \subset (\Delta^{\sqcup})^*$ be the set of all paths in $\mathbb{A}^{\sqcup}$ starting with the initial state 0 and including the empty path ε. For $w \in A$, $Z(w)$ denotes the final state of the path and $Z(\varepsilon) := 0$. Formally the prefix closed language A and the function $Z : A \to \mathbb{N}_0^Q$ is defined inductively by $\varepsilon \in A$, $Z(\varepsilon) := 0$, and if $w \in A$ with $Z(w) = f$ and $(f, a, g) \in \Delta^{\sqcup}$ then $w(f, a, g) \in A$ and $Z(w(f, a, g)) := g$. Let $\alpha' : (\Delta^{\sqcup})^* \to \Sigma^*$ be the homomorphism with $\alpha'((f, a, g)) = a$ for $(f, a, g) \in \Delta^{\sqcup}$, and let $\alpha := \alpha'_{|A}$. Hence $w \in A$ is an accepting path of a word $u \in \Sigma^*$ iff $Z(w) = 0$ and $\alpha(w) = u$. We denote the language accepted by $\mathbb{A}^{\sqcup}$ with $L(\mathbb{A}^{\sqcup})$.

Theorem 3.

$$P^{\sqcup} \subset L(\mathbb{A}^{\sqcup}) \tag{22a}$$

$$L(\mathbb{A}^{\sqcup}) \subset P^{\sqcup} \tag{22b}$$

The automaton $\mathbb{A}^{\sqcup}$ accepts the language $P^{\sqcup}$.

Proof. Together with the alphabet Σ we now consider four pairwise disjoint alphabets $\tilde{\Sigma}$, $\mathring{\Sigma}$, $\bar{\Sigma}$, $\tilde{\tilde{\Sigma}}$ and a homomorphism $\wedge : \hat{\Sigma}^* \to \Sigma^*$ with $\hat{\Sigma} := \tilde{\Sigma} \cup \mathring{\Sigma} \cup \bar{\Sigma} \cup \tilde{\tilde{\Sigma}}$ and $\wedge(\tilde{a}) := \wedge(\mathring{a}) := \wedge(\bar{a}) := \wedge(\tilde{\tilde{a}}) := a$ for each $a \in \Sigma$.

For words $u \in P \subset \Sigma^+$ the four alphabets are used to characterise start-, inner-, end-, or start-end letters of u. Let therefore $\hat{P} := \wedge^{-1}(P) \cap [\tilde{\Sigma}\mathring{\Sigma}^*\bar{\Sigma} \cup \tilde{\tilde{\Sigma}}] \subset \hat{\Sigma}^+$. Since $\varepsilon \notin P$, $\wedge_{|\hat{P}} : \hat{P} \to P$ is a bijection.

According to the definitions of Σ_t, $\tau_t^{\mathbb{N}}$ and $\Theta^{\mathbb{N}}$ we now consider $\hat{\Sigma}_t := \tilde{\Sigma}_t \cup \mathring{\Sigma}_t \cup \bar{\Sigma}_t \cup \tilde{\tilde{\Sigma}}_t$ for $t \in \mathbb{N}$ and $\hat{\tau}_t^{\mathbb{N}} : \hat{\Sigma}_{\mathbb{N}}^* \to \hat{\Sigma}^*$ and $\hat{\Theta}^{\mathbb{N}} : \hat{\Sigma}_{\mathbb{N}}^* \to \hat{\Sigma}^*$, with $\hat{\Sigma}_{\mathbb{N}} := \bigcup_{s \in \mathbb{N}} \hat{\Sigma}_s$. Therewith now

$$P^{\sqcup} = \Theta^{\mathbb{N}}\Big(\bigcap_{t \in \mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})\Big) = \wedge\Big[\hat{\Theta}^{\mathbb{N}}\Big(\bigcap_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\hat{P} \cup \{\varepsilon\})\Big)\Big] = \wedge(\hat{P}^{\sqcup}).$$

We can now show two relations between $\mathrm{pre}[\bigcap_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\hat{P} \cup \{\varepsilon\})] = \bigcap_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})(\mathrm{pre}(\hat{P}))$ and A that prove Theorem 3. We first show

Proposition 1. *For each* $x \in \bigcap_{t\in\mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(pre(\hat{P}))$ *there exists* $y \in A$ *such that* $\alpha(y) = \wedge(\hat{\Theta}^{\mathbb{N}}(x))$ *and* $Z(y)(q) = \#(\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x))) = q$ *and* $\hat{\tau}_t^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}\})$ *for each* $q \in Q$*, where* $\#(X)$ *denotes the cardinality of the set* X*.*

Taking into account that $x \in \bigcap_{t\in\mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\hat{P} \cup \{\varepsilon\})$ iff $x \in \bigcap_{t\in\mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\text{pre}(\hat{P}))$ and $\hat{\tau}_t^{\mathbb{N}}(x) \in \hat{P} \cup \{\varepsilon\}$ for each $t \in \mathbb{N}$ then from Proposition 1 it directly follows that $P^{\sqcup} \subset L(\mathbb{A}^{\sqcup})$.

Proof of Proposition 1 by induction:
Induction base. With $y = \varepsilon$, Proposition 1 holds for $x = \varepsilon$.
Induction step.
Let $x' = x\hat{a}_s \in \text{pre}[\bigcap_{t\in\mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\hat{P} \cup \{\varepsilon\})]$ with $\hat{a}_s \in \hat{\Sigma}_s$.
Then also $x \in \text{pre}[\bigcap_{t\in\mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\hat{P} \cup \{\varepsilon\})]$ and by induction hypothesis it exists an $y \in A$ such that $\alpha(y) = \wedge(\hat{\Theta}^{\mathbb{N}}(x))$ and $Z(y)(q) = \#(\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x))) = q$ and $\hat{\tau}_t^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}\})$ for each $q \in Q$.
According to the union $\hat{\Sigma} := \tilde{\Sigma} \cup \mathring{\Sigma} \cup \bar{\Sigma} \cup \bar{\tilde{\Sigma}}$ we now have to consider four cases:

Case 1: $\hat{a}_s \in \tilde{\Sigma}_s$

Then $\delta(q_0, \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s))) = p$, it exists $b \in \Sigma$ such that $\delta(p, b)$ is defined, $\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s) \notin \hat{P} \cup \{\varepsilon\}$ and $\hat{\tau}_s^{\mathbb{N}}(x) = \varepsilon$.
Because of the first two statements, we have $(Z(y), \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)), Z(y) + 1_p) \in \tilde{\Delta}$, which implies $y' = y(Z(y), \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)), Z(y) + 1_p) \in A$ and
$\alpha(y') = \alpha(y) \wedge (\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)) = \wedge(\hat{\Theta}^{\mathbb{N}}(x)) \wedge (\hat{\Theta}^{\mathbb{N}}(\hat{a}_s)) = \wedge(\hat{\Theta}^{\mathbb{N}}(x'))$.
Since $\hat{\tau}_s^{\mathbb{N}}(x) = \varepsilon$, $\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s) \notin \hat{P} \cup \{\varepsilon\}$ and $\hat{\tau}_t^{\mathbb{N}}(\hat{a}_s) = \varepsilon$ for $t \in \mathbb{N} \setminus \{\varepsilon\}$, it holds:
$\#(\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s))) = q'$ and $\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s) \notin \hat{P} \cup \{\varepsilon\}\}) =$
$= \#(\{t \in \mathbb{N} \setminus \{s\} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s))) = q'$ and $\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s) \notin \hat{P} \cup \{\varepsilon\}\}) + \#(\{t \in \{s\} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s))) = q'$ and $\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s) \notin \hat{P} \cup \{\varepsilon\}\}) =$
$= \#(\{t \in \mathbb{N} \setminus \{s\} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x))) = q'$ and $\hat{\tau}_t^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}\}) + \#(\{t \in \{s\} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(\hat{a}))) = q'\}) =$
$= \#(\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x))) = q'$ and $\hat{\tau}_t^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}\}) + 1_p(q') = Z(y)(q') + 1_p(q')$ for each $q' \in Q$. This completes the induction step for case 1.

Case 2: $\hat{a}_s \in \mathring{\Sigma}_s$

Then $\delta(q_0, \wedge(\hat{\tau}_s^{\mathbb{N}}(x))) = q$, $\delta(q, \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s))) = p$, and it exists $b \in \Sigma$ such that $\delta(p, b)$ is defined, $\hat{\tau}_s^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}$ and $\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s) \notin \hat{P} \cup \{\varepsilon\}$.
Because of the first three statements, $Z(y) \geqslant 1_q$ and $(Z(y), \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)), Z(y) + 1_p - 1_q) \in \mathring{\Delta}$. As in case 1 it follows $y' = y(Z(y), \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)), Z(y) + 1_p - 1_q) \in A$ and $\alpha(y') = \alpha(y) \wedge (\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)) = \wedge(\hat{\Theta}^{\mathbb{N}}(x)) \wedge (\hat{\Theta}^{\mathbb{N}}(\hat{a}_s)) = \wedge(\hat{\Theta}^{\mathbb{N}}(x'))$.
Since $\hat{\tau}_s^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}$ and $\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s) \notin \hat{P} \cup \{\varepsilon\}$ analogue to case 1 it holds $\#(\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s))) = q'$ and $\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s) \notin \hat{P} \cup \{\varepsilon\}\}) = \#(\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x))) = q'$ and $\hat{\tau}_t^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}\}) - 1_q(q') + 1_p(q')$ for each $q' \in Q$. This completes the induction step for case 2.

Case 3: $\hat{a}_s \in \bar{\Sigma}_s$

Then $\delta(q_0, \wedge(\hat{\tau}_s^{\mathbb{N}}(x))) = q$, $\delta(q, \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s))) \in F$, $\hat{\tau}_s^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}$ and $\hat{\tau}_s^{\mathbb{N}}(x\hat{a}_s) \in \hat{P}$.
Because of the first two statements, $Z(y) \geqslant 1_q$ and $(Z(y), \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)), Z(y) -$

$1_q) \in \bar{\Delta}$. As in case 1 it follows $y' = y(Z(y), \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)), Z(y) - 1_q) \in A$ and $\alpha(y') = \alpha(y) \wedge (\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)) = \wedge(\hat{\Theta}^{\mathbb{N}}(x)) \wedge (\hat{\Theta}^{\mathbb{N}}(\hat{a}_s)) = \wedge(\hat{\Theta}^{\mathbb{N}}(x'))$.
Since $\hat{\tau}_s^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}$ and $\hat{\tau}_s^{\mathbb{N}}(x\hat{a}_s) \in \hat{P}$ analogue to case 2 it holds $\#(\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s))) = q'$ and $\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s) \notin \hat{P} \cup \{\varepsilon\}\}) = Z(y)(q') - 1_q(q') + 0(q')$ for each $q' \in Q$. This completes the induction step for case 3.

Case 4: $\hat{a}_s \in \bar{\tilde{\Sigma}}_s$
Then $\delta(q_0, \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s))) \in F$, $\hat{\tau}_s^{\mathbb{N}}(x) = \varepsilon$ and $\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s) \in \hat{P}$. Because of the first statement, $(Z(y), \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)), Z(y)) \in \bar{\tilde{\Delta}}$.
As in case 1 it follows $y' = y(Z(y), \wedge(\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)), Z(y)) \in A$ and $\alpha(y') = \alpha(y) \wedge (\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s)) = \wedge(\hat{\Theta}^{\mathbb{N}}(x)) \wedge (\hat{\Theta}^{\mathbb{N}}(\hat{a}_s)) = \wedge(\hat{\Theta}^{\mathbb{N}}(x'))$.
Since $\hat{\tau}_s^{\mathbb{N}}(x) = \varepsilon$ and $\hat{\tau}_s^{\mathbb{N}}(\hat{a}_s) \in \hat{P}$ analogue to case 1 and case 3 it holds $\#(\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s))) = q'$ and $\hat{\tau}_t^{\mathbb{N}}(x\hat{a}_s) \notin \hat{P} \cup \{\varepsilon\}\}) = Z(y)(q') + 0(q')$ for each $q' \in Q$. This completes the induction step for case 4 and the proof of Proposition 1.

For the proof of $L(\mathbb{A}^{\sqcup}) \subset P^{\sqcup}$ (22b) we now show

Proposition 2. *For each $y \in A$ exists $x \in \bigcap_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(pre(\hat{P}))$ such that $\alpha(y) = \wedge(\hat{\Theta}^{\mathbb{N}}(x))$ and $Z(y)(q) = \#(\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x))) = q$ and $\hat{\tau}_t^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}\})$ for each $q \in Q$.*

As in Proposition 1 from Proposition 2 follows $L(\mathbb{A}^{\sqcup}) \subset P^{\sqcup}$ (22b).
Proof of Proposition 2 by induction:
Induction base. With $x = \varepsilon$, Proposition 2 holds for $y = \varepsilon$.
Induction step.
Let $y' = y(Z(y), a, g) \in A$ with $(Z(y), a, g) \in \Delta^{\sqcup}$. Then also $y \in A$ and by induction hypothesis exists an $x \in \bigcap_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\text{pre}(\hat{P}))$ such that $\alpha(y) = \wedge(\hat{\Theta}^{\mathbb{N}}(x))$ and $Z(y)(q) = \#(\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}_t^{\mathbb{N}}(x))) = q$ and $\hat{\tau}_t^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}\})$ for each $q \in Q$.

According to the (not necessarily disjoint) union $\Delta^{\sqcup} = \tilde{\Delta} \cup \mathring{\Delta} \cup \bar{\Delta} \cup \bar{\tilde{\Delta}}$ we have to consider four cases:

Case 1: $(Z(y), a, g) \in \tilde{\Delta}$
Then $g = Z(y) + 1_p$ with $\delta(q_0, a) = p$ and it exists $b \in \Sigma$ such that $\delta(p, b)$ is defined and it exists $s \in \mathbb{N}$ such that $\hat{\tau}_s^{\mathbb{N}}(x) = \varepsilon$. Let now $\tilde{a}_s \in \tilde{\Sigma}_s$ with $\wedge(\hat{\tau}_s^{\mathbb{N}}(\tilde{a}_s)) = a$. Then $\tilde{a} = \hat{\tau}_s^{\mathbb{N}}(\tilde{a}_s) \in \text{pre}(\hat{P}) \setminus \hat{P}$ and hence $x' = x\tilde{a}_s \in \bigcap_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\text{pre}(\hat{P}))$. The necessary properties of x' for the induction step can now be shown as in case 1 of Proposition 1.

Case 2: $(Z(y), a, g) \in \mathring{\Delta}$
Then $Z(y) \geqslant 1_q$, $g = Z(y) + 1_p - 1_q$, $\delta(q, a) = p$ and it exists $b \in \Sigma$ such that $\delta(p, b)$ is defined . Since $Z(y) \geqslant 1_q$ it exists $s \in \mathbb{N}$ such that $\delta(q_0, \wedge(\hat{\tau}_s^{\mathbb{N}}(x))) = q$ and $\hat{\tau}_s^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}$. Let $\mathring{a}_s \in \mathring{\Sigma}_s$ with $\wedge(\hat{\tau}_s^{\mathbb{N}}(\mathring{a}_s)) = a$. Since $\delta(q, a) = p$ and it exists $b \in \Sigma$ such that $\delta(p, b)$ is defined, it holds $\hat{\tau}_s^{\mathbb{N}}(x\mathring{a}_s) \in \text{pre}(\hat{P}) \setminus (\hat{P} \cup \{\varepsilon\})$.
Therewith $x' = x\mathring{a}_s \in \bigcap_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\text{pre}(\hat{P}))$ and as in case 2 of Proposition 1 the necessary properties of x' for the induction step can be shown.

Case 3: $(Z(y), a, g) \in \bar{\Delta}$

Then $Z(y) \geqslant 1_q$, $g = Z(y) - 1_q$ and $\delta(q, a) \in F$. Since $Z(y) \geqslant 1_q$ it exists $s \in \mathbb{N}$ such that $\delta(q_0, \wedge(\hat{\tau}_s^{\mathbb{N}}(x))) = q$ and $\hat{\tau}_s^{\mathbb{N}}(x) \notin \hat{P} \cup \{\varepsilon\}$. Let $\bar{a}_s \in \bar{\Sigma}_s$ with $\wedge(\hat{\tau}_s^{\mathbb{N}}(\bar{a}_s)) = a$. $\delta(q, a) \in F$ implies $\hat{\tau}_s^{\mathbb{N}}(x\bar{a}_s) \in \hat{P}$ and hence $x' = x\bar{a}_s \in \bigcap\limits_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\text{pre}(\hat{P}))$. As in case 3 of Proposition 1 the necessary properties of x' for the induction step can now be shown.

Case 4: $(Z(y), a, g) \in \tilde{\bar{\Delta}}$

Then $\delta(q_0, a) \in F$. Let $s \in \mathbb{N}$ with $\hat{\tau}_s^{\mathbb{N}}(x) = \varepsilon$ and $\tilde{\bar{a}}_s \in \tilde{\bar{\Sigma}}_s$ with $\wedge(\hat{\tau}_s^{\mathbb{N}}(\tilde{\bar{a}}_s)) = a$. $\delta(q_0, a) \in F$ implies $\hat{\tau}_s^{\mathbb{N}}(\tilde{\bar{a}}_s) \in \hat{P}$ and hence $x' = x\tilde{\bar{a}}_s \in \bigcap\limits_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\text{pre}(\hat{P}))$. As in case 4 of Proposition 1 the necessary properties of x' for the induction step can now be shown, which completes the proof of Proposition 2.

From Propositions 1 and 2 follows $L(\mathbb{A}^{\sqcup\!\sqcup}) = P^{\sqcup\!\sqcup}$. This completes the proof of Theorem 3. □

5. DETERMINISM IN SHUFFLE AUTOMATA

Related to the automaton $\mathbb{A}^{\sqcup\!\sqcup}$ we introduced another form of structured representations that we will now refer to as *structured $\wedge$-representations*. For $P \subset \Sigma^+$ is was mentioned that $\wedge_{|\hat{P}} : \hat{P} \to P$ is a bijection.

Therefore $P^{\sqcup\!\sqcup} = \Theta^{\mathbb{N}}[\bigcap\limits_{t \in \mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})] = (\wedge \circ \hat{\Theta}^{\mathbb{N}})[\bigcap\limits_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\hat{P} \cup \{\varepsilon\})]$

For $x \in \Sigma^*$ let $\hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})}(x) := (\wedge \circ \hat{\Theta}^{\mathbb{N}})^{-1}(x) \cap [\bigcap\limits_{t \in \mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\hat{P} \cup \{\varepsilon\})]$. It is the set of all structured $\wedge$-representations of x.

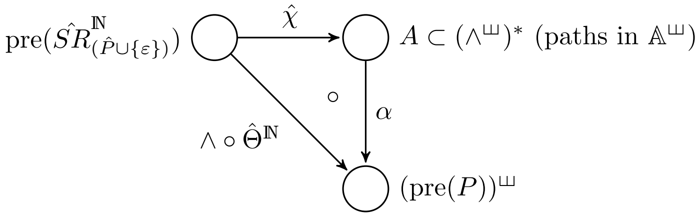
It is easy to see that the mapping $\hat{a}_t \mapsto a_t$ for $t \in T$ defines a bijection from $\hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})}(x)$ onto $SR^{\mathbb{N}}_{(P \cup \{\varepsilon\})}(x)$.

Regarding $\text{pre}(\hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})}(x))$ and $\text{pre}(SR^{\mathbb{N}}_{(P \cup \{\varepsilon\})}(x))$ the relation above defines a mapping that is surjective but not necessarily injective.

Let e.g. $P = \{ab, abc\}$, then $\tilde{a}_1\bar{b}_1 \in \hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})}(ab) \subset \text{pre}(\hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})}(ab))$ and $\tilde{a}_1\mathring{b}_1 \in \text{pre}(\hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})}(ab))$ that are both mapped to $a_1b_1 \in SR^{\mathbb{N}}_{(P \cup \{\varepsilon\})}(ab) \subset \text{pre}(SR^{\mathbb{N}}_{(P \cup \{\varepsilon\})}(ab))$ by the mapping above.

In Theorem 3 a relation between the elements of $\text{pre}(\hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})}(x))$ and the paths in $\mathbb{A}^{\sqcup\!\sqcup}$ was established. Let therefore $\hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})} := \bigcup\limits_{x \in P^{\sqcup\!\sqcup}} \hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})}(x)$.

Then $\wedge \circ \hat{\Theta}^{\mathbb{N}} : \text{pre}(\hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})}) \to (\text{pre}(P))^{\sqcup\!\sqcup}$ is a surjective mapping. The construction in (22a) defines a mapping $\hat{\chi} : \text{pre}(\hat{SR}^{\mathbb{N}}_{(\hat{P} \cup \{\varepsilon\})}) \to A$ with $\alpha \circ \hat{\chi} = \wedge \circ \hat{\Theta}^{\mathbb{N}}$, and so



Definition 7. $\mathbb{A}^{\sqcup\!\sqcup}$ *is called* deterministic on $w \in (\text{pre}(P))^{\sqcup\!\sqcup}$, *iff* $\#(\alpha^{-1}(x)) = 1$ *for each* $x \in pre(w)$. *In that case, we consider* $\alpha^{-1}(x)$ *as an element of* A *instead of a subset of* A.

Theorem 3 provides a relation between the states of the shuffle automaton $\mathbb{A}^{\sqcup\!\sqcup}$ and the structured $\wedge$-representations. With an additional determinism condition the following theorem provides an analog relation for structured representations.

Theorem 4. *Let* $\mathbb{A}^{\sqcup\!\sqcup}$ *be deterministic on* $w \in (pre(P))^{\sqcup\!\sqcup}$, S *a countable index set and* $w'' \in SR^S_{pre(P)}(w)$, *then*

$$Z[\alpha^{-1}(w)](q) = \#(\{s \in S \mid \delta(q_0, \tau^S_s(w'')) = q \text{ and } \tau^S_s(w'') \notin P \cup \{\varepsilon\}\}) \quad \text{for each } q \in Q, \qquad (23a)$$

and

$$y \notin P \text{ for each } y \in pre(\tau^S_t(w'')) \setminus \{\tau^S_t(w'')\} \subset pre(P) \text{ and } t \in S. \qquad (23b)$$

Proof. (23a):
Since $\mathbb{A}^{\sqcup\!\sqcup}$ is deterministic on w, according to (22a)
$Z[\alpha^{-1}(w)](q) = \#\{t \in \mathbb{N} \mid \delta(q_0, \wedge(\hat{\tau}^{\mathbb{N}}_t(x'))) = q$ and $\hat{\tau}^{\mathbb{N}}_t(x') \notin \hat{P} \cup \{\varepsilon\}\})$ for each $q \in Q$ and each $x' \in \bigcap_{t\in\mathbb{N}} (\hat{\tau}^{\mathbb{N}}_t)^{-1}(\text{pre}(\hat{P}))$ with $w = \wedge(\hat{\Theta}^{\mathbb{N}}(x'))$.

We now extend the homomorphisms $\wedge : \hat{\Sigma}^* \to \Sigma^*$ defined in the proof of Theorem 3 to a homomorphisms $\wedge : (\hat{\Sigma} \cup \hat{\Sigma}_{\mathbb{N}})^* \to (\Sigma \cup \Sigma_{\mathbb{N}})^*$ such that the mapping $\hat{a}_s \to a_s$ for $a_s \in \Sigma_s$ and $s \in \mathbb{N}$ is included.
So the restriction $\wedge_{|\text{pre}(\hat{SR}^{\mathbb{N}}_{(\hat{P}\cup\{\varepsilon\})})} : \text{pre}(\hat{SR}^{\mathbb{N}}_{(\hat{P}\cup\{\varepsilon\})}) \to \text{pre}(SR^{\mathbb{N}}_{(P\cup\{\varepsilon\})})$ is the mapping introduced in the context of defining $\hat{SR}^{\mathbb{N}}_{(\hat{P}\cup\{\varepsilon\})}$ which is surjective but not necessarily injective. With this definition we now have $\wedge(\hat{\tau}^{\mathbb{N}}_t(x')) = \tau^{\mathbb{N}}_t(\wedge(x'))$ for each $x' \in \bigcap_{t\in\mathbb{N}} (\hat{\tau}^{\mathbb{N}}_t)^{-1}(\text{pre}(\hat{P}))$ and $t \in \mathbb{N}$. For such x' and t holds $\hat{\tau}^{\mathbb{N}}_t(x') = \varepsilon \Leftrightarrow \tau^{\mathbb{N}}_t(\wedge(x')) = \varepsilon$ and $\hat{\tau}^{\mathbb{N}}_t(x') \in \hat{P} \Rightarrow \tau^{\mathbb{N}}_t(\wedge(x')) \in P$, but not necessarily the reverse implication.

If $\tau^{\mathbb{N}}_t(\wedge(x')) \in P$ and $\hat{\tau}^{\mathbb{N}}_t(x') \notin \hat{P}$ for a $t \in \mathbb{N}$, then $\hat{\tau}^{\mathbb{N}}_t(x') = \tilde{a}_t$ and $\tau^{\mathbb{N}}_t(\wedge(x')) = a_t$ with $a_t \in \Sigma_t$ or $\hat{\tau}^{\mathbb{N}}_t(x') = \hat{u}_t\mathring{a}_t$ and $\tau^{\mathbb{N}}_t(\wedge(x')) = \wedge(\hat{u}_t)a_t$ with $a_t \in \Sigma_t$ and $\hat{u}_t \in \tilde{\Sigma}_t\mathring{\Sigma}^*_t$.
So x' can be decomposed into $x' = v_1\tilde{a}_tv_2$ respectively $x' = v_1\mathring{a}_tv_2$ with $v2 \in \Sigma^*_{\mathbb{N}\setminus\{t\}}$. Because $a_t \in P$ respectively $\wedge(\hat{u}_t)a_t \in P$, $v_1\tilde{a}_t$ respectively $v_1\mathring{a}_t$ as well as $v_1\tilde{\bar{a}}_t$ respectively $v_1\bar{a}_t$ are prefixes of structured $\wedge$-representations with $\wedge(\hat{\Theta}^{\mathbb{N}}(v_1\tilde{a}_t)) =$

$\wedge(\hat{\Theta}^{\mathbb{N}}(v_1\tilde{\tilde{a}}_t)) \in \mathrm{pre}(w)$ respectively $\wedge(\hat{\Theta}^{\mathbb{N}}(v_1\mathring{a}_t)) = \wedge(\hat{\Theta}^{\mathbb{N}}(v_1\bar{a}_t)) \in \mathrm{pre}(w)$. However $\hat{\chi}(v_1\tilde{a}_t) \neq \hat{\chi}(v_1\tilde{\tilde{a}}_t)$ respectively $\hat{\chi}(v_1\mathring{a}_t) \neq \hat{\chi}(v_1\bar{a}_t)$, which contradicts the determinism of $\mathbb{A}^{\sqcup\!\sqcup}$ on w.
Therefore the implication $\tau_t^{\mathbb{N}}(\wedge(x')) \in P \Rightarrow \hat{\tau}_t^{\mathbb{N}}(x') \in \hat{P}$ holds, and so
$Z[\alpha^{-1}(w)](q) = \#(\{t \in \mathbb{N} \mid \delta(q_0, \tau_t^{\mathbb{N}}(\wedge(x'))) = q$ and $\tau_t^{\mathbb{N}}(\wedge(x')) \notin P \cup \{\varepsilon\}\})$ for each $q \in Q$.

Because for each $w' \in SR^{\mathbb{N}}_{\mathrm{pre}(P)} = \mathrm{pre}(SR^{\mathbb{N}}_{(P\cup\{\varepsilon\})}) = \bigcap_{t\in\mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(\mathrm{pre}(P))$ there exists $x' \in \mathrm{pre}(\hat{SR}^{\mathbb{N}}_{(\hat{P}\cup\{\varepsilon\})}) = \bigcap_{t\in\mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\mathrm{pre}(\hat{P}))$ with $w' = \wedge(x')$, it holds
$Z[\alpha^{-1}(w)](q) = \#(\{t \in \mathbb{N} \mid \delta(q_0, \tau_t^{\mathbb{N}}(w')) = q$ and $\tau_t^{\mathbb{N}}(w') \notin P \cup \{\varepsilon\}\})$ for each $q \in Q$ and $w' \in SR^{\mathbb{N}}_{\mathrm{pre}(P)}(w')$.
Let now $\iota : S \to \mathbb{N}$ be a bijection.
According to Lemma 4 $\nu_\iota : SR^S_{\mathrm{pre}(P)}(w) \to SR^{\mathbb{N}}_{\mathrm{pre}(P)}(w)$ is an isomorphism with $\tau^{\mathbb{N}}_{\iota(s)} \circ \nu_\iota = \tau_s^S$ for each $s \in S$. For $w'' \in SR^S_{\mathrm{pre}(P)}(w)$ let now $w' = \nu_\iota(w'') \in SR^{\mathbb{N}}_{\mathrm{pre}(P)}(w)$ and hence $\tau^{\mathbb{N}}_{\iota(s)}(w') = \tau_s^S(w'')$.

From this it follows that $\{t \in \mathbb{N} \mid \delta(q_0, \tau_t^{\mathbb{N}}(w')) = q$ and $\tau_t^{\mathbb{N}}(w') \notin P \cup \{\varepsilon\}\} = \{\iota(s) \in \mathbb{N} \mid \delta(q_0, \tau^{\mathbb{N}}_{\iota(s)}(w')) = q$ and $\tau^{\mathbb{N}}_{\iota(s)}(w') \notin P \cup \{\varepsilon\}\} = \{s \in S \mid \delta(q_0, \tau_s^S(w'')) = q$ and $\tau_s^S(w'') \notin P \cup \{\varepsilon\}\}$, which proves (23a).

(23b):
If $\tau_t^S(w'') = \varepsilon$, then the proposition holds because $\mathrm{pre}(\varepsilon) \setminus \{\varepsilon\} = \emptyset$. Therefore let $\tau_t^S(w'') \neq \varepsilon$.
Because of the assumption $\varepsilon \notin P$, the proposition holds for $y = \varepsilon$. Therefore let $y \neq \varepsilon$. As in the proof of (23a), it is sufficient to prove the proposition for $S = \mathbb{N}$. As in the proof of (23a), there exists $x' \in \bigcap_{t\in\mathbb{N}} (\hat{\tau}_t^{\mathbb{N}})^{-1}(\mathrm{pre}(\hat{P}))$ such that $w'' = \wedge(x')$ and $\tau_t^{\mathbb{N}}(w'') = \wedge(\hat{\tau}_t^{\mathbb{N}}(x'))$. Furthermore there exists $y' \in \mathrm{pre}(\hat{\tau}_t^{\mathbb{N}}(x')) \setminus \{\varepsilon, \hat{\tau}_t^{\mathbb{N}}(x')\}$ and $y = \wedge(y')$. Hence $y' = \tilde{a}_t$ with $a_t \in \Sigma_t$ or $y' = \hat{u}_t\mathring{a}_t$ with $a_t \in \Sigma_t$ and $\hat{u}_t \in \tilde{\Sigma}_t\mathring{\Sigma}_t^*$. As in the proof of (23a), the assumption $y \in P$ now contradicts the determinism of $\mathbb{A}^{\sqcup\!\sqcup}$ on w. □

6. SUFFICIENT CONDITION FOR SELF-SIMILARITY

Theorem 4 together with the remarks following Lemma 5 contribute considerably to a sufficient condition for (14). Therefore the following definition is reasonable.

Definition 8. *A prefix-closed language $L \subset \Sigma^*$ is* based deterministically on a phase *$P \subset \Sigma^+$ w.r.t. $\mathbb{P}$, if L is based on P and the deterministic automaton $\mathbb{P}$ accepts P, so that $\mathbb{P}^{\sqcup\!\sqcup}$ is deterministic on each $w \in L \subset (\mathrm{pre}(P))^{\sqcup\!\sqcup}$.*

If L is accepted by a deterministic automaton $\mathbb{L}$, then L is based deterministically on P w.r.t. $\mathbb{P}$, iff L is based on P and the product automaton [Sakarovitch 2009] of $\mathbb{L}$ and $\mathbb{P}^{\sqcup\!\sqcup}$ is deterministic.

We now continue the remarks following Lemma 5. Additionally we assume that $SF \subset \Phi^*$ is based deterministically on $PF \subset \Phi^+$ w.r.t. $\mathbb{PF}$. At this, let $\mathbb{PF}$ be a deterministic automaton accepting PF, so that $\mathbb{PF}^{\sqcup\!\sqcup}$ is deterministic on each $u \in SF$.

Now by Theorem 4 $Z[\alpha^{-1}(u)]$ formally and unambiguously describes "how a cooperation partner is involved in several phases".

Furthermore, let $\emptyset \neq K' \subset K$ and let K be finite.

For $\bar{\varphi}^K(w) \in SF$ and $y \in SR^{K\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^K(w))$ according to Theorem 4 we have $Z[\alpha^{-1}(\bar{\varphi}^K(w))](q) = \#(\{(s,t) \in K \times \mathbb{N} \mid \delta(q_0, \tau^{K\times\mathbb{N}}_{(s,t)}(y)) = q \text{ and } \tau^{K\times\mathbb{N}}_{(s,t)}(y) \notin PF \cup \{\varepsilon\}\})$ for each $q \in Q$. If additionally $\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w)) \in SF$, then by (21) $\Pi^{K\times\mathbb{N}}_{K'\times\mathbb{N}}(y) \in SR^{K'\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w)))$, and also according to Theorem 4 it holds

$$\begin{aligned} Z[\alpha^{-1}(\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w)))](q) =& \#(\{(s,t) \in K' \times \mathbb{N} \mid \delta(q_0, \tau^{K'\times\mathbb{N}}_{(s,t)}(\Pi^{K\times\mathbb{N}}_{K'\times\mathbb{N}}(y))) = q \\ & \text{and } \tau^{K'\times\mathbb{N}}_{(s,t)}(\Pi^{K\times\mathbb{N}}_{K'\times\mathbb{N}}(y)) \notin PF \cup \{\varepsilon\}\}) \\ =& \#(\{(s,t) \in K' \times \mathbb{N} \mid \delta(q_0, \tau^{K\times\mathbb{N}}_{(s,t)}(y)) = q \\ & \text{and } \tau^{K\times\mathbb{N}}_{(s,t)}(y) \notin PF \cup \{\varepsilon\}\}) \\ \leq& Z[\alpha^{-1}(\bar{\varphi}^K(w))](q) \text{ for each } q \in Q. \end{aligned} \tag{24}$$

Now let $wa_k \in [\bigcap_{s\in K} (\bar{\varphi}^K_s)^{-1}(SF)] \cap (\bar{\varphi}^K)^{-1}(SF) \subset \Phi^*_K$ with $a_k \in \Phi_k$ and $k \in K$. According to (20) there exists $y' \in SR^{K\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^K(wa_k))$ such that $wa_k = \Theta^{K\times\mathbb{N}}_K(y')$. This implies

$$y' = ya_{(k,n)} \text{ with } y \in SR^{K\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^K(w)) \text{ as well as } a_{(k,n)} \in \Phi_{(k,n)} \text{ with } n \in \mathbb{N}. \tag{25}$$

Therewith holds $\tau^{K\times\mathbb{N}}_{(k,n)}(y') = \tau^{K\times\mathbb{N}}_{(k,n)}(y)a \in \text{pre}(PF)$ and $a = \tau^{K\times\mathbb{N}}_{(k,n)}(a_{(k,n)}) \in \Phi$. If therefore $a \notin \text{pre}(PF)$, then this implies $\tau^{K\times\mathbb{N}}_{(k,n)}(y) \neq \varepsilon$ and $\tau^{K\times\mathbb{N}}_{(k,n)}(y) \in \text{pre}(\tau^{K\times\mathbb{N}}_{(k,n)}(y')) \setminus \{\tau^{K\times\mathbb{N}}_{(k,n)}(y')\}$. Since $\bar{\varphi}^K(wa_k) \in SF$ and according to the hypothesis that $\mathbb{PF}^{\sqcup}$ is deterministic on $\bar{\varphi}^K(wa_k)$, as well as according to (23b) it holds

$$\varepsilon \neq \tau^{K\times\mathbb{N}}_{(k,n)}(y) \notin PF \tag{26}$$

Now (24) and (26) provide the formal base for a sufficient condition for (14).

Let $\mathbb{PF} = (\Phi, Q, \delta, q_0, F)$ be a deterministic automaton that accepts PF and let $\mathbb{SF} = (\Phi, Q_{SF}, \delta_{SF}, q_{SF0})$ be a deterministic automaton that accepts SF. If SF is deterministically based on PF w.r.t. $\mathbb{PF}$, then holds

Theorem 5. *If for each $(q_{SF}, f) \in Q_{SF} \times \mathbb{N}_0^Q$ and $(q'_{SF}, f') \in Q_{SF} \times \mathbb{N}_0^Q$ for which exists $u, u' \in SF \cap (pre(PF))^{\sqcup}$ such that $q_{SF} = \delta_{SF}(q_{SF0}, u)$, $q'_{SF} = \delta_{SF}(q_{SF0}, u')$, $f = Z[\alpha^{-1}(u)]$, $f' = Z[\alpha^{-1}(u')]$ and for which $f \geq f'$ the following holds:*

$$\{a \in \Phi \cap pre(PF) \mid \delta_{SF}(q_{SF}, a) \text{ is defined }\} \subset \{a \in \Phi \cap pre(PF) \mid \delta_{SF}(q'_{SF}, a) \text{ is defined }\} \tag{27a}$$

and for each $q \in Q$ *with* $f'(q) > 0$ *is*

$$\{a \in \Phi \setminus \mathit{pre}(PF) \mid \delta(q,a) \text{ and } \delta_{SF}(q_{SF},a) \text{ are defined }\} \subset \{a \in \Phi \setminus \mathit{pre}(PF) \mid \delta(q,a) \text{ and } \delta_{SF}(q'_{SF},a) \text{ are defined }\} \quad (27\text{b})$$

then follows

$$\hat{\varphi}^K_{K'}[[\bigcap_{s\in K} (\bar{\varphi}^K_s)^{-1}(SF)] \cap (\bar{\varphi}^K)^{-1}(SF)] \subset (\bar{\varphi}^{K'})^{-1}(SF) \text{ for all } \emptyset \neq K' \subset K.$$

Proof. We show $\hat{\varphi}^K_{K'}(v) \in (\bar{\varphi}^{K'})^{-1}(SF)$ by induction on $v \in [\bigcap_{s\in K} (\bar{\varphi}^K_s)^{-1}(SF)] \cap (\bar{\varphi}^K)^{-1}(SF) \subset \Phi^*_K$, as $[\bigcap_{s\in K} (\bar{\varphi}^K_s)^{-1}(SF)] \cap (\bar{\varphi}^K)^{-1}(SF)$ is prefix closed.

Induction base. For $v = \varepsilon$ holds $\hat{\varphi}^K_{K'}(v) = \varepsilon \in (\bar{\varphi}^{K'})^{-1}(SF)$.

Induction step.

Let $v = wa_k$ with $a_k \in \Phi_k$ and $k \in K$, then $w \in [\bigcap_{s\in K} (\bar{\varphi}^K_s)^{-1}(SF)] \cap (\bar{\varphi}^K)^{-1}(SF)$, and by induction hypothesis $\hat{\varphi}^K_{K'}(w) \in (\bar{\varphi}^{K'})^{-1}(SF)$.

Case 1: $k \notin K'$

Then $\hat{\varphi}^K_{K'}(v) = \hat{\varphi}^K_{K'}(w) \in (\bar{\varphi}^{K'})^{-1}(SF)$.

Case 2: $k \in K'$

$\bar{\varphi}^K(w) \in SF$ because of $w \in [\bigcap_{s\in K} (\bar{\varphi}^K_s)^{-1}(SF)] \cap (\bar{\varphi}^K)^{-1}(SF)$.

According to (25) there exists $y \in SR^{K\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^K(w))$ and according to induction hypothesis $\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w)) \in SF$.

From (24) and (21) it follows $Z[\alpha^{-1}(\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w)))] \leq Z[\alpha^{-1}(\bar{\varphi}^K(w))]$ and $\Pi^{K\times\mathbb{N}}_{K'\times\mathbb{N}}(y) \in SR^{K'\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w)))$.

Let now $u := \bar{\varphi}^K(w)$ and $u' := \bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w))$, then $u \in SF$ and $u' \in SF$. The existence of $y \in SR^{K\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^K(w))$ and $\Pi^{K\times\mathbb{N}}_{K'\times\mathbb{N}}(y) \in SR^{K'\times\mathbb{N}}_{\text{pre}(PF)}(\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w)))$ implies $u \in (\text{pre}(PF))^{\sqcup}$ and $u' \in (\text{pre}(PF))^{\sqcup}$.

Let now additionally $q_{SF} := \delta_{SF}(q_{SF0}, u)$, $q'_{SF} := \delta_{SF}(q_{SF0}, u')$, $f := Z[\alpha^{-1}(u)]$ and $f' := Z[\alpha^{-1}(u')]$, then $f \geq f'$.

$v \in [\bigcap_{s\in K} (\bar{\varphi}^K_s)^{-1}(SF)] \cap (\bar{\varphi}^K)^{-1}(SF)$ implies $\bar{\varphi}^K(v) \in SF$ and $\bar{\varphi}^K(v) = \bar{\varphi}^K(w)a$ with $a \in \Phi$. So $\delta_{SF}(q_{SF}, a)$ is defined.

$k \in K'$ implies $\hat{\varphi}^K_{K'}(v) = \hat{\varphi}^K_{K'}(w)a_k$, and so $\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(v)) = \bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w))a$.

To complete the induction step it remains to prove, that $\delta_{SF}(q'_{SF}, a)$ is defined.

For case 2.1: $a \in \Phi \cap \text{pre}(PF)$ this follows from the precondition (27a).

For case 2.2: $a \in \Phi \setminus \text{pre}(PF)$, it follows according to (26) that there exists $n \in \mathbb{N}$ with $\varepsilon \neq \tau^{K\times\mathbb{N}}_{(k,n)}(y) \notin PF$. Since $\tau^{K\times\mathbb{N}}_{(k,n)}(y)a \in \text{pre}(PF)$ (25) it exists $q \in Q$ such that $\delta(q_0, \tau^{K\times\mathbb{N}}_{(k,n)}(y)) = q$ and $\delta(q,a)$ is defined. Because of (24) for this q holds $f'(q) = Z[\alpha^{-1}(\bar{\varphi}^{K'}(\hat{\varphi}^K_{K'}(w)))](q) = \#(\{(s,t) \in K' \times \mathbb{N} \mid \delta(q_0, \tau^{K\times\mathbb{N}}_{(s,t)}(y)) = q \text{ and } \tau^{K\times\mathbb{N}}_{(s,t)}(y) \notin PF \cup \{\varepsilon\}\}) \geq 1$. From precondition (27b) it follows that $\delta_{SF}(q'_{SF}, a)$ is defined, which completes the proof of Theorem 5.

□

Corollary 1. *If all prerequisites of Theorem 5 and additionally those with respect to SG and PG are fulfilled, then* $\Pi^{IK}_{I'K'}(\mathcal{L}_{IK}) = \mathcal{L}_{I'K'}$ *for* $I' \times K' \subset I \times K$.

The hypotheses for Theorem 5 can be checked at the product automaton of $\mathbb{SF}$ and $\mathbb{PF}^{\sqcup\!\sqcup}$, if it is finite. If $\mathbb{PF}$ and $\mathbb{SF}$ are finite automata, then the reachable part of the product automaton can be constructed step by step (reachability analysis). If the product automaton is finite, this procedure terminates. Therefore finiteness of the product automaton can be verified by a semi-algorithm.

7. CONCLUSIONS AND FUTURE WORK

The main result of this paper is a sufficient condition for self-similarity of uniformly parameterised cooperations. Under certain regularity restrictions this condition can be verified by a semi-algorithm.

It is well known that dynamic system properties are divided into safety and liveness properties [Alpern and Schneider 1985]. Safety properties can be formalised by prefix closed languages. For abstractions defined by alphabetic language homomorphisms it is easy to see that an abstract system satisfies a safety property iff the concrete system satisfies a corresponding safety property. So our notion of self-similarity is compatible with safety properties.

Concerning liveness properties this does not hold in general. In [Nitsche and Ochsenschläger 1996] a property of homomorphisms is given that implies a similar relation between liveness properties of an abstract and a concrete system w.r.t. a modified satisfyability notion. Based on that framework we will investigate liveness aspects of uniformly parameterised cooperations in a forthcoming paper. Another topic of interest is the generalisation of this paper to n-sided cooperations.

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